Diploma Programme
Programme du diplôme
Programa del Diploma

Mathematics: analysis and approaches
Higher level
Paper 3

14 November 2025

Zone A afternoon | **Zone B** afternoon | **Zone C** afternoon

1 hour 15 minutes

Instructions to candidates

- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Answer all the questions in the answer booklet provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches HL formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[55 marks]**.

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Answer **all** questions in the answer booklet provided. Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 26]

The following question explores features of composed trigonometric functions, such as $\sin(\sin x)$**,** $\sin(\sin(\sin x))$.

Suppose $S_n(x)$ denotes the function $\sin x$ composed within itself $n-1$ times, defined for $n \geq 1$, $n \in \mathbb{Z}^+$, where $0 \leq x \leq 2\pi$.

For example, $S_1(x) = \sin x$ and $S_2(x) = \sin(\sin x)$ where $0 \leq x \leq 2\pi$.

(a) On the same axes, sketch and label the graphs of $y = S_1(x)$ and $y = S_2(x)$. On your sketch, show the values of the intercepts with the axes. [4]

(b) Determine the maximum value of

(i) $S_1(x)$; [1]

(ii) $S_2(x)$; [1]

(iii) $S_3(x)$. [1]

(c) Find the least value of n for which the maximum value of $S_n(x)$ is less than 0.6. [3]

Consider the graph of $y = S_2(x)$.

(d) By considering the equation $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$, show that there are exactly two points of zero gradient, one at $x = \frac{\pi}{2}$ and one at $x = \frac{3\pi}{2}$. [6]

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(Question 1 continued)

The derivative $S'_n(x) = \frac{\mathrm{d}}{\mathrm{d}x}(S_n(x))$ can be expressed as a product of cosine functions, as follows:

$$S'_n(x) = \cos(S_{n-1}(x))\cos(S_{n-2}(x)) \ldots \cos(S_1(x))\cos x.$$

(e) Hence, show that $S'_3(x) = \cos(\sin(\sin x))\cos(\sin x)\cos x$. [1]

(f) Use mathematical induction to prove that for all $n \in \mathbb{Z}^+$

$S'_n(x) = \cos(S_{n-1}(x))\cos(S_{n-2}(x)) \ldots \cos(S_1(x))\cos x.$ [6]

(g) Use l'Hôpital's rule to show that $\lim_{x\to 0}\frac{S_n(x)}{x} = 1$, for $n \in \mathbb{Z}^+$. [3]

2. [Maximum mark: 29]

The following question uses Maclaurin series to investigate approximations of mathematical constants and the accuracies of such approximations.

(a) Given $|x| < 1$, find the sum to infinity of the geometric series $1 - x^2 + x^4 - x^6 + \ldots$. [2]

(b) Hence, use integration to show that the Maclaurin series of $\arctan x$ may be expressed as $\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$. [3]

(c) Using $x = \frac{1}{\sqrt{3}}$ and the first **three** (non-zero) terms of the Maclaurin series of $\arctan x$, find an approximation for π to three decimal places. [3]

The Maclaurin series of $\arctan x$ is an example of an alternating series, *ie* a series where consecutive terms are positive and negative. Consider the following theorem.

Theorem: For alternating series with terms of decreasing magnitude, the error obtained in using a finite number of terms is less than or equal to the absolute value of the next term in the sequence.

Using the theorem, the maximum error in using the first three (non-zero) terms as an approximation to $\arctan x$ is given by $\left|-\frac{x^7}{7}\right|$. In other words, $\left|\arctan x - \left(x - \frac{x^3}{3} + \frac{x^5}{5}\right)\right| \le \left|-\frac{x^7}{7}\right|$.

(d) Determine how many (non-zero) terms of the series would need to be used, such that the error in approximating $\arctan\left(\frac{1}{\sqrt{3}}\right)$ is less than 0.0001. [3]

(e) By using integration by parts, show that $\int_0^{\frac{1}{\sqrt{3}}} \arctan x \, dx = \frac{\pi}{6\sqrt{3}} - \frac{1}{2}\ln\frac{4}{3}$. [4]

(f) Determine the value of $\int_0^{\frac{1}{\sqrt{3}}} \left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7}\right) dx$, giving your answer to six decimal places. [2]

(g) Use the results in parts (e) and (f) to find an approximation for π. Give your answer to four decimal places. [2]

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(Question 2 continued)

$$\int_0^{\frac{1}{\sqrt{3}}} \left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots \right) \mathrm{d}x$$ may be considered as the sum of alternating terms.

Hence, $$\int_0^{\frac{1}{\sqrt{3}}} \left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots \right) \mathrm{d}x = \int_0^{\frac{1}{\sqrt{3}}} x \, \mathrm{d}x + \int_0^{\frac{1}{\sqrt{3}}} \left(-\frac{x^3}{3} \right) \mathrm{d}x + \int_0^{\frac{1}{\sqrt{3}}} \left(\frac{x^5}{5} \right) \mathrm{d}x + \int_0^{\frac{1}{\sqrt{3}}} \left(-\frac{x^7}{7} \right) \mathrm{d}x + \ldots .$$

$$\int_0^{\frac{1}{\sqrt{3}}} \arctan x \, \mathrm{d}x$$ is approximated using the sum of the first four definite integrals.

(h) Verify that the theorem used in part (d) holds in this case. [5]

Suppose that the maximum error in approximating $\int_0^{\frac{1}{\sqrt{3}}} \arctan x \, \mathrm{d}x$ is required to be at most 1×10^{-6}.

(i) Determine the smallest number of (non-zero) terms of the Maclaurin series for $\arctan x$ that should be used. [5]