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# Mathematics: analysis and approaches

## Higher level

### Paper 3

14 November 2025

Zone A afternoon | Zone B afternoon | Zone C afternoon

1 hour 15 minutes

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#### Instructions to candidates

- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Answer all the questions in the answer booklet provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches HL formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[55 marks]**.

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Answer **all** questions in the answer booklet provided. Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 26]

**This question asks you to investigate lines normal to curves of the form  $y = \frac{k^2}{x}$ .**

The curve  $H$  has equation  $y = \frac{1}{x}$  where  $x \in \mathbb{R}, x \neq 0$ .

(a) A line  $N$  is normal to  $H$  at  $x = t$ .

(i) Show that the gradient of  $N$  is  $t^2$ . [2]

(ii) Hence, show that the equation of  $N$  is  $y = t^2x + \frac{1}{t} - t^3$ . [1]

(b) The equation for  $N$  given in part (a)(ii) is of the form  $y = mx + c$ .

(i) Show that either  $c = \frac{1}{\sqrt{m}}(1 - m^2)$  or  $c = \frac{1}{\sqrt{m}}(m^2 - 1)$ . [4]

(ii) Determine the set of values of  $m$  for which there exists at least one line normal to  $H$ . [1]

(c) Hence, or otherwise, determine the set of values of  $m$  for which there exists exactly

(i) one line normal to  $H$ ; [1]

(ii) two lines normal to  $H$ . [2]

(d) On the same set of axes, sketch

(i) the curve  $H$ ; [1]

(ii) for an appropriate value of  $m$ , two lines that satisfy the result found in part (c)(ii). Clearly indicate the point at which each line is normal to  $H$ . [2]

You are **not** required to state the equations of these lines nor determine where they intersect  $H$  or the coordinate axes.

**(This question continues on the following page)**

**(Question 1 continued)**

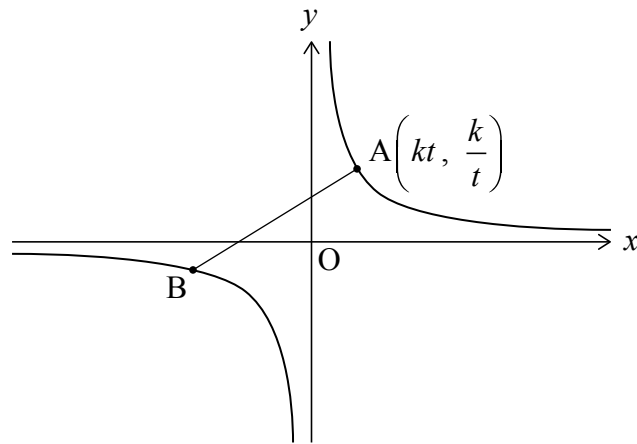
The curve  $F$  has equation  $y = \frac{k^2}{x}$  where  $x \in \mathbb{R}, x \neq 0$  and  $k \in \mathbb{R}, k \neq 0$ .

The point  $A\left(kt, \frac{k}{t}\right)$ , where  $t \in \mathbb{R}, t \neq \pm 1$ , lies on  $F$ .

The line normal to  $F$  at  $A$  intersects  $F$  again at point  $B$ .

The line segment  $[AB]$  is shown in the following diagram.

**diagram not to scale**



(e) The equation of the line normal to  $F$  at  $A$  is given by  $y = t^2x - kt^3 + \frac{k}{t}$ .

(i) Show that the  $x$ -coordinates of  $A$  and  $B$  satisfy the quadratic equation

$$x^2 - k\left(t - \frac{1}{t^3}\right)x - \frac{k^2}{t^2} = 0. \quad [3]$$

(ii) Hence, by considering either the sum or product of the roots of this quadratic equation, or otherwise, determine the coordinates of  $B$ . [3]

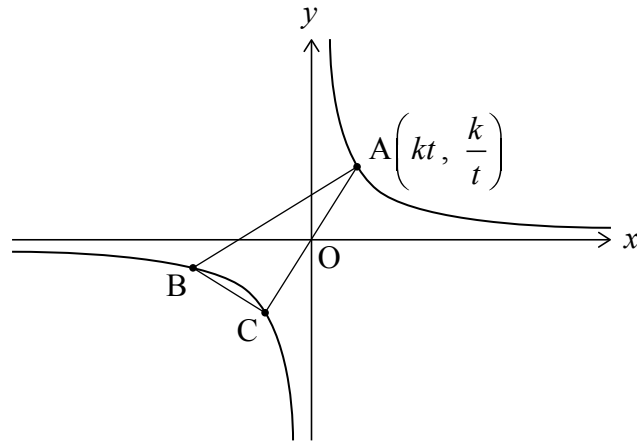
**(This question continues on the following page)**

**(Question 1 continued)**

From  $A$ , the line passing through the origin  $O$  intersects  $F$  again at point  $C$ .

Points  $A$ ,  $B$  and  $C$  form triangle  $ABC$  as shown in the following diagram.

**diagram not to scale**



(f) Prove that  $\hat{B}CA$  is a right angle.

[6]

2. [Maximum mark: 29]

**In this question researchers are trying to find the most accurate model to use when modelling a population of wolves.**

Historically, a population of wolves in an area had a stable size of 200. After some years of disruption, the population was reduced to 40 wolves. At this point, the area became a protected space and the population began to grow again.

Researchers in the area wish to model the size of the wolf population,  $x$ , as a function of  $t$ , where  $t$  is the time, in years, since the area became protected.

(a) Initially, the researchers consider using the logistic model

$$x = \frac{L}{1 + Ce^{-kt}}, \text{ where } L, C, k \in \mathbb{R}^+.$$

The researchers decide to let  $L = 200$ .

(i) State the assumption being made in assuming  $L = 200$ . [1]

At  $t = 0$ , the population of wolves is 40.

(ii) Find the value of  $C$ . [2]

At  $t = 5$ , the population of wolves is found to have increased to 70.

(iii) Find the value of  $k$ . [2]

(iv) Use your model to predict the size of the wolf population in the area 10 years after it became protected. Give your answer correct to the nearest whole number. [2]

(b) An alternative model for population growth is called the Gompertz model. When applied by the researchers to the wolf population, this model satisfies the differential equation

$$\frac{dx}{dt} = ax \ln\left(\frac{200}{x}\right), \text{ } a \in \mathbb{R}^+.$$

(i) Write down the value of  $\frac{dx}{dt}$  when  $x = 200$ . [1]

(ii) Interpret your answer to part (b)(i) in context. [1]

**(This question continues on the following page)**

**(Question 2 continued)**

Consider the function  $f(x) = \ln(\ln 200 - \ln x)$ , where  $0 < x < 200$ .

(iii) Show that  $f'(x) = \frac{-1}{x \ln\left(\frac{200}{x}\right)}$ . [2]

(iv) Hence, use separation of variables to show that the general solution of

$$\frac{dx}{dt} = ax \ln\left(\frac{200}{x}\right), \text{ where } 0 < x < 200,$$

can be written as

$$\ln x = \ln 200 - Ae^{-at},$$

where  $A$  is an arbitrary positive constant. [5]

(v) Use the size of the wolf population at  $t = 0$  to find the value of  $A$ .  
Give your answer in the form  $A = \ln p$ , where  $p \in \mathbb{Z}^+$ . [2]

(vi) Use the size of the wolf population at  $t = 5$ , given in part (a), to show that  $a = 0.0855$ , correct to three significant figures. [2]

(vii) Use the Gompertz model to predict the size of the wolf population at  $t = 10$ .  
Give your answer correct to the nearest whole number. [3]

After 10 years, the wolf population is measured and is found to be 85.

(c) Comment on the predictions made by the two models. [1]

By tracking individual wolves, the researchers find that about 3% of the wolf population emigrate from the protected area each year.

They decide to adapt the Gompertz model to allow for this. The new model will satisfy the differential equation

$$\frac{dx}{dt} = 0.0855x \ln\left(\frac{200}{x}\right) - 0.03x.$$

(d) (i) Use Euler's method, with a step size of 0.5 years and an initial value of  $x_0 = 70$  when  $t = 5$ , to find an estimate for the size of the wolf population when  $t = 10$ .  
Give your answer correct to the nearest whole number. [4]

(ii) Comment on your answer. [1]