Diploma Programme
Programme du diplôme
Programa del Diploma

Mathematics: analysis and approaches
Higher level
Paper 3

21 May 2025

Zone A afternoon | **Zone B** afternoon | **Zone C** afternoon

1 hour 15 minutes

Instructions to candidates

- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Answer all the questions in the answer booklet provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches HL formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[55 marks]**.

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Answer **all** questions in the answer booklet provided. Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 27]

This question asks you to explore self-composite linear functions, of the form $f(x) = mx + c$, for varying values of m.

A function composed with itself is called a self-composite function.

For a function f, the function composition with itself is given by $(f \circ f)(x) = f\left(f(x)\right)$.

Let f^n denote the nth composition of f with itself where $f^n(x) = \underbrace{\left(f \circ f \circ \cdots \circ f\right)}_{n \text{ times}}(x)$.

Hence, for example, $f^2(x) = (f \circ f)(x)$ and $f^3(x) = (f \circ f \circ f)(x) = f\left(f\left(f(x)\right)\right)$.

Consider the linear function $f(x) = mx + c$, where $x \in \mathbb{R}$ and $m, c \in \mathbb{R}$.

(a) Show that

(i) $f^2(x) = m^2x + c(1 + m)$; [3]

(ii) $f^3(x) = m^3x + c(1 + m + m^2)$. [2]

(b) (i) Write down an expression for $f^4(x)$. [1]

(ii) Suggest a similar expression for $f^n(x)$, $n \in \mathbb{Z}^+$. [2]

(iii) By using your expression from part (b)(ii), or otherwise, find an expression in terms of n for $f^n(x)$ when $m = 1$. [3]

(c) For $m \neq 1$, use mathematical induction to prove that

$$f^n(x) = m^n x + c\left(\frac{1 - m^n}{1 - m}\right) \text{ for } n \in \mathbb{Z}^+. \quad [8]$$

(This question continues on the following page)

(Question 1 continued)

Consider $f^n(x) = m^n x + c\left(\frac{1-m^n}{1-m}\right)$ where $-1 < m < 1$.

(d) As $n \to \infty$, the family of graphs $y = f^n(x)$ approaches the graph of a straight line, L. Determine the equation of L, giving your answer in terms of c and m. [4]

Consider $f^n(x) = m^n x + c\left(\frac{1-m^n}{1-m}\right)$ where $m = -1$.

(e) (i) Show that $f^n(x) = -x + c$ when n is odd. [2]

(ii) Find an expression for $f^n(x)$ when n is even. [2]

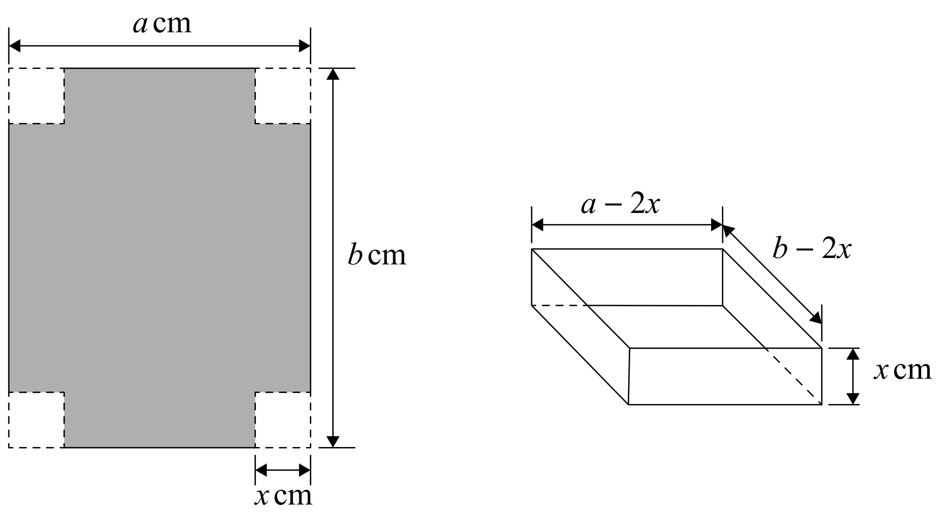
2. [Maximum mark: 28]

This question asks you to consider open-topped boxes formed by cutting square corners from an $a \times b$ sheet of cardboard and folding the sides. In particular, you are asked to investigate conditions for which the maximum volume of a box is an integer.

A rectangular sheet of cardboard measures a cm by b cm where $a < b$.

A square of side length x cm is cut from each corner and the resulting sheet of cardboard is then folded to form an open-topped box with a rectangular base.

This is shown in the following diagrams.



The volume, $V \text{cm}^3$, of a box is given by $V = x(a-2x)(b-2x)$ where $0 \le x \le \frac{a}{2}$.

(a) Write down the value of V when

(i) $x = 0$; [1]

(ii) $x = \frac{a}{2}$. [1]

(b) Show that $V = 4x^3 - 2(a+b)x^2 + abx$. [2]

(c) Show that the only solutions of $\frac{dV}{dx} = 0$ are $x = \frac{(a+b) \pm \sqrt{a^2 - ab + b^2}}{6}$. [6]

(This question continues on the following page)

(Question 2 continued)

In parts (d) and (e), the following three results are given.

$$x = \frac{(a+b)+\sqrt{a^2-ab+b^2}}{6} \text{ lies outside the interval } 0 \le x \le \frac{a}{2}.$$

$$x = \frac{(a+b)-\sqrt{a^2-ab+b^2}}{6} \text{ lies inside the interval } 0 \le x \le \frac{a}{2}.$$

$$\sqrt{a^2-ab+b^2} > 0.$$

Let $x_m = \dfrac{(a+b)-\sqrt{a^2-ab+b^2}}{6}$.

(d) Use $\dfrac{d^2V}{dx^2}$ to show that there is a local maximum of V at $x = x_m$. [5]

(e) Use your answers to parts (a) and (d) to explain why x_m will give the maximum volume, V_m, of the box. [1]

In parts (f) to (h), consider a method for generating sets of positive integer values for x_m and hence V_m.

Consider the case where $a = 2st - s^2$, $b = t^2 - s^2$, and $s, t \in \mathbb{Z}^+$.

(f) Given that $a < b$, show that $t > 2s$. [2]

It is given that $\sqrt{a^2-ab+b^2} = s^2 - st + t^2$. You are **not** required to show this result.

(g) By substituting for each of a, b and $\sqrt{a^2-ab+b^2}$ in terms of s and t, show that $x_m = \dfrac{s(t-s)}{2}$. [3]

(h) (i) Given that $a, b \in \mathbb{Z}^+$, use $V_m = x_m(a-2x_m)(b-2x_m)$ to explain why s being even implies that V_m is an integer. [2]

(ii) State another condition on s and a corresponding condition on t, that implies that both x_m and V_m are integers. [1]

(iii) Hence, or otherwise, find an integer value of x_m and the corresponding integer value of V_m. [4]