Diploma Programme
Programme du diplôme
Programa del Diploma

Mathematics: analysis and approaches
Higher level
Paper 3

21 May 2025

Zone A afternoon | **Zone B** afternoon | **Zone C** afternoon

1 hour 15 minutes

Instructions to candidates

- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Answer all the questions in the answer booklet provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches HL formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[55 marks]**.

2225 – 7113

7 pages

Answer **all** questions in the answer booklet provided. Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 28]

The following question explores features of a family of curves. The family is then linked to a homogeneous differential equation.

Consider the curve given by $y = \frac{x\left(x^2 - 16\right)}{x^2 + 16}$.

(a) (i) Sketch the curve of y for $-10 \leq x \leq 10$. [1]

(ii) State the coordinates of the points where the curve crosses the x-axis. [1]

(iii) State the coordinates of the local maximum point and the coordinates of the local minimum point. [2]

(b) State whether the function $f(x) = \frac{x\left(x^2 - 16\right)}{x^2 + 16}$ is odd, even or neither. Justify your answer. [2]

Now consider the general curve given by $y = \frac{x\left(x^2 - A\right)}{x^2 + A}$, where A is a positive constant and $x \in \mathbb{R}$.

(c) Given $f\left(x\right) = \frac{x\left(x^2 - A\right)}{x^2 + A}$, prove that $f'\left(\sqrt{A}\right)$ is independent of A. [4]

(d) (i) Show that $x - \frac{2Ax}{x^2 + A} \equiv \frac{x\left(x^2 - A\right)}{x^2 + A}$. [2]

(ii) Hence, determine the equation of the oblique asymptote to the curve. [1]

(iii) Write down the coordinates of a point on the curve where the oblique asymptote is parallel to the tangent to the curve at that point. [1]

(This question continues on the following page)

(Question 1 continued)

Now consider the differential equation $x^2 \frac{\mathrm{d}y}{\mathrm{d}x} = x(x+y) - y^2$, where $x \neq 0$, $y \neq \pm x$.

Using the substitution $y = vx$, the differential equation can be written as $x\frac{\mathrm{d}v}{\mathrm{d}x} = 1 - v^2$.

(e) Using partial fractions, show that $\int \frac{1}{1-v^2}\,\mathrm{d}v = \frac{1}{2}\ln\left|\frac{A(1+v)}{1-v}\right|$, where A is a positive constant. [5]

(f) Hence, show that a solution to the original differential equation may be expressed in the form $x^2 = \left|\frac{A(x+y)}{x-y}\right|$, where A is a positive constant. [5]

Now consider only the case where $\frac{A(x+y)}{x-y} > 0$.

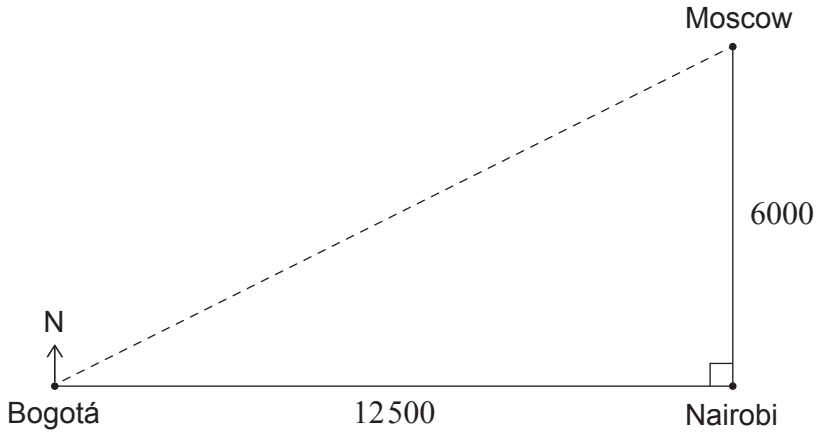
(g) Show that a solution to the original differential equation is $y = \frac{x(x^2 - A)}{x^2 + A}$. [4]

2. [Maximum mark: 27]

The following question compares the distance and direction between cities on a flat surface to the distance and direction between cities on a sphere.

Consider a model where the cities of Bogotá, Moscow, and Nairobi lie on a flat surface. In this model, Nairobi is $6000\,\text{km}$ due south of Moscow and Bogotá is $12\,500\,\text{km}$ due west of Nairobi, as shown in the following diagram.

diagram not to scale



(a) (i) Find the distance from Bogotá to Moscow. [2]

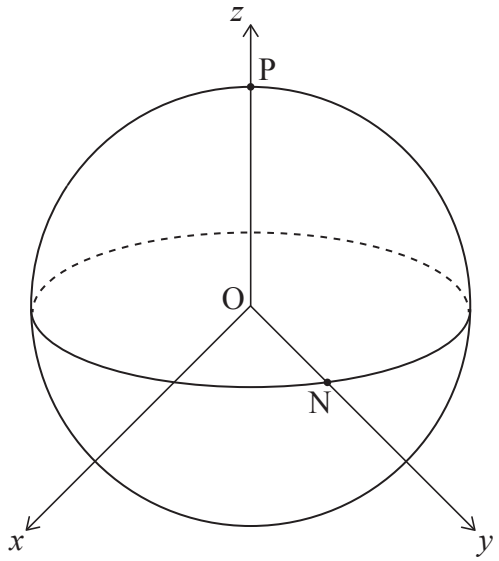
(ii) Find the bearing of Moscow from Bogotá. Give your answer in degrees. [3]

(This question continues on the following page)

(Question 2 continued)

In reality, these three cities lie on the curved surface of the Earth which will change the distances and directions found in part (a).

Now consider a curved model using a coordinate system (x, y, z) with its origin, O, at the centre of the Earth. The units of this system are thousands of kilometres and the Earth is modelled as a sphere with radius $6000\,\mathrm{km}$. The North Pole, P, lies on the z-axis, and Nairobi, N, is modelled as being on the equator and lying on the y-axis.



P has position vector $\overrightarrow{\mathrm{OP}} = \boldsymbol{p} = \begin{pmatrix} 0 \\ 0 \\ 6 \end{pmatrix}$ and N has position vector $\overrightarrow{\mathrm{ON}} = \boldsymbol{n} = \begin{pmatrix} 0 \\ 6 \\ 0 \end{pmatrix}$.

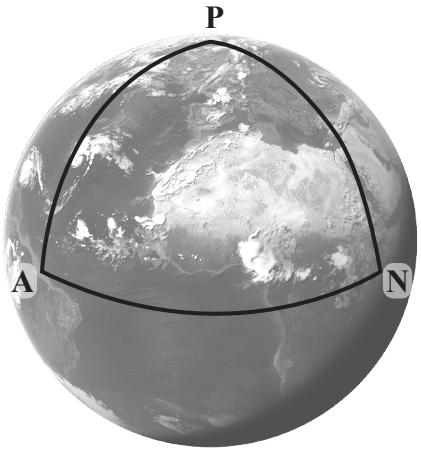
(b) (i) Use the scalar product to find the angle between $\boldsymbol{p}$ and $\boldsymbol{n}$. [2]

(ii) Show that the distance between P and N along the arc from P to N is $3000\pi\,\mathrm{km}$. [2]

(This question continues on the following page)

(Question 2 continued)

Point A, which is also on the equator, has position vector $\boldsymbol{a} = \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix}$ as shown in the following diagram.



P, N and A, and the arcs connecting them, form a spherical triangle.

The angle at vertex A is defined as the angle between the vectors $\boldsymbol{a} \times \boldsymbol{p}$ and $\boldsymbol{a} \times \boldsymbol{n}$.

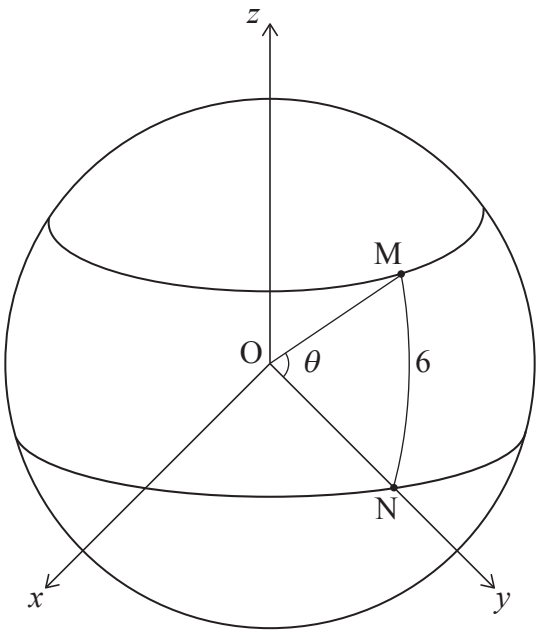
(c) (i) Find the vector $\boldsymbol{a} \times \boldsymbol{p}$. [2]

(ii) Show that the angle at vertex A in the spherical triangle is $90°$. [3]

(This question continues on the following page)

(Question 2 continued)

Moscow, M, has position vector $\overrightarrow{\mathrm{OM}} = \boldsymbol{m} = \begin{pmatrix} 0 \\ 6\cos\theta \\ 6\sin\theta \end{pmatrix}$, as shown in the following diagram.



The shortest distance between two points on the sphere lies along an arc of a circle on the sphere with centre O. In this model the shortest distance from Moscow to Nairobi is $6000\,\mathrm{km}$.

(d) Show that $\theta = 57.3°$, correct to three significant figures. [2]

Bogotá, B, is west of Nairobi and has position vector $\overrightarrow{\mathrm{OB}} = \boldsymbol{b} = \begin{pmatrix} 6\sin 120^\circ \\ 6\cos 120^\circ \\ 0 \end{pmatrix}$.

(e) Find the shortest distance from Bogotá to Moscow on the sphere. [5]

The bearing from B to M is defined as the angle at vertex B in the spherical triangle containing B, M and P. It is given that $\boldsymbol{b} \times \boldsymbol{p} = \begin{pmatrix} 36\cos 120^\circ \\ -36\sin 120^\circ \\ 0 \end{pmatrix}$.

(f) Using the method from part (c), find the bearing from Bogotá to Moscow. [6]

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References:

2(c) Google Maps/Google Earth. Data from Data SIO, NOAA, U.S. Navy, NGA, GEBCOLandsat / Copernicus, IBCAO, U.S. Geological Survey, PGC/NASA. Imagery from 14/12/2015. Image available at: https://earth.google.com/web/@12.01529518,-18.56070747,-158.39383184a,23597813.93249989d,30.00008083y,359.99981502h,0t,0r/data=CgRCAggBOgMKATBCAggASg0I_____ARAA.