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Mathematics: analysis and approaches

Higher level

Paper 3

21 May 2025

Zone A afternoon | Zone B afternoon | Zone C afternoon

1 hour 15 minutes

Instructions to candidates

- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Answer all the questions in the answer booklet provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches HL formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[55 marks]**.

Answer **all** questions in the answer booklet provided. Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 23]

This question asks you to use polynomial functions to model some situations in probability.

Two unbiased tetrahedral (four-sided) dice with faces labelled 1, 2, 3 and 4 are thrown and the scores recorded.

The random variable M denotes the maximum of these two scores.

The probability distribution of M is given in the following table.

m	1	2	3	4
$P(M = m)$	$\frac{1}{16}$	$\frac{3}{16}$	$\frac{5}{16}$	$\frac{7}{16}$

(a) Find $E(M)$. [2]

An alternative way to represent the probability distribution of M is to use a polynomial

function, G , where $G(t) = \sum_{m=1}^4 P(M = m)t^m$.

Hence, for the distribution of M , $G(t) = \frac{1}{16}t + \frac{3}{16}t^2 + \frac{5}{16}t^3 + \frac{7}{16}t^4$.

(b) Find $G(1)$. [1]

(c) (i) Find $G'(t)$. [2]

(ii) Hence, show that $G'(1) = E(M)$. [3]

(This question continues on the following page)

(Question 1 continued)

A bag contains two red balls and three yellow balls.

Two balls are selected at random without replacement from the bag.

The random variable X denotes the total number of red balls selected.

The probability distribution of X can be represented by the polynomial function, G_X , where

$$G_X(t) = \sum_{x=0}^2 P(X = x)t^x.$$

- (d) Show that $G_X(t) = \frac{3}{10} + \frac{3}{5}t + \frac{1}{10}t^2$, making it clear how the coefficients of $G_X(t)$ have been determined. [5]

An unbiased coin and a biased coin are tossed.

The probability of obtaining a tail on the biased coin is p .

The random variable Y denotes the total number of tails obtained from tossing both coins.

The probability distribution of Y can be represented by the polynomial function, G_Y , where

$$G_Y(t) = \sum_{y=0}^2 P(Y = y)t^y.$$

- (e) Given that the coefficient of t^2 in $G_Y(t)$ is $\frac{1}{3}$, find

(i) the value of p ; [2]

(ii) an expression for $G_Y(t)$. [4]

The random variable Z denotes the sum of the total number of red balls selected, X , and the total number of tails obtained from tossing both coins, Y .

The probability distribution of Z can be represented by the function, G_Z , where

$$G_Z(t) = G_X(t)G_Y(t).$$

- (f) For random variable Z , it can be shown that $G_Z'(1) = E(Z)$.

Use this result to find $E(Z)$. [4]

2. [Maximum mark: 32]

Informally, the curvature of a curve can be thought of as the amount by which the curve deviates from being a straight line. In this question, you will investigate the curvature of a variety of functions.

Consider any function f that can be differentiated twice.

The curvature, k , of any function f is defined by $k(x) = \frac{|f''(x)|}{\left(1 + (f'(x))^2\right)^{\frac{3}{2}}}$.

Consider the family of linear functions $g(x) = mx + c$, where $x \in \mathbb{R}$ and $m, c \in \mathbb{R}$.

(a) Show that $k(x) = 0$ for this family of linear functions. [2]

Consider the family of quadratic functions $h(x) = ax^2 + bx + c$ for $x \in \mathbb{R}$, where $a \in \mathbb{R}$, $a \neq 0$ and $b, c \in \mathbb{R}$.

For this family of quadratic functions, it is given that

$$k(x) = \frac{2|a|}{\left(1 + (2ax + b)^2\right)^{\frac{3}{2}}} \text{ and}$$

$$k'(x) = -\frac{12a|a|(2ax + b)}{\left(1 + (2ax + b)^2\right)^{\frac{5}{2}}}.$$

The maximum value of $k(x)$ is denoted by k_{\max} .

(b) (i) By solving $k'(x) = 0$, find the value of x where k_{\max} occurs.
You are **not** required to justify that this value of x leads to a maximum. [1]

(ii) Determine an expression for k_{\max} , in terms of a only. [2]

(iii) State the value of $\lim_{x \rightarrow \infty} k(x)$ and explain briefly the significance of this result. [2]

(iv) Consider the quadratic functions

$$p(x) = -2x^2 + 2x - 10 \text{ where } x \in \mathbb{R} \text{ and}$$

$$q(x) = 2x^2 + 5x + 25 \text{ where } x \in \mathbb{R}.$$

State which one of the following statements is true and justify your answer.

A. k_{\max} of $p > k_{\max}$ of q

B. k_{\max} of $p < k_{\max}$ of q

C. k_{\max} of $p = k_{\max}$ of q

[2]

(This question continues on the following page)

(Question 2 continued)

Consider the function $v(x) = \ln x$, where $x \in \mathbb{R}$, $x > 0$.

For v , it is given that

$$k(x) = \frac{x}{(1+x^2)^{\frac{3}{2}}} \text{ and}$$

$$k'(x) = \frac{1-2x^2}{(1+x^2)^{\frac{5}{2}}}.$$

- (c) (i) Determine the exact value of x where k_{\max} occurs.

You are **not** required to justify that this value of x leads to a maximum. [2]

- (ii) Show that $k_{\max} = \frac{2\sqrt{3}}{9}$. [4]

Consider the function $w(x) = e^x$, where $x \in \mathbb{R}$.

For w , it is given that

$$k(x) = \frac{e^x}{(1+e^{2x})^{\frac{3}{2}}} \text{ and}$$

$$k'(x) = \frac{e^x(1-2e^{2x})}{(1+e^{2x})^{\frac{5}{2}}}.$$

- (d) (i) Show that $k_{\max} = \frac{2\sqrt{3}}{9}$. [5]

- (ii) Suggest a reason why v and w have the same k_{\max} . [1]

Consider a family of curves $y = \sqrt{r^2 - x^2}$, where $-r < x < r$, $y > 0$ and r is a positive constant.

- (e) (i) Show that $\frac{d^2y}{dx^2} = -\frac{r^2}{y^3}$. [6]

- (ii) Hence, show that the curvature, k , is constant for this family of curves. [5]