# Mathematics: analysis and approaches
# Higher level
# Paper 3

21 May 2025

**Zone A** afternoon | **Zone B** afternoon | **Zone C** afternoon

1 hour 15 minutes

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**Instructions to candidates**

- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Answer all the questions in the answer booklet provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches HL formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[55 marks]**.



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Answer **all** questions in the answer booklet provided. Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

**1.** [Maximum mark: 23]

**This question asks you to use polynomial functions to model some situations in probability.**

Two unbiased tetrahedral (four-sided) dice with faces labelled $1$, $2$, $3$ and $4$ are thrown and the scores recorded.

The random variable $M$ denotes the maximum of these two scores.

The probability distribution of $M$ is given in the following table.

| $m$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $\mathrm{P}(M=m)$ | $\frac{1}{16}$ | $\frac{3}{16}$ | $\frac{5}{16}$ | $\frac{7}{16}$ |

(a) Find $\mathrm{E}(M)$. [2]

An alternative way to represent the probability distribution of $M$ is to use a polynomial function, $G$, where $G(t)=\sum_{m=1}^{4}\mathrm{P}(M=m)t^m$.

Hence, for the distribution of $M$, $G(t)=\frac{1}{16}t+\frac{3}{16}t^2+\frac{5}{16}t^3+\frac{7}{16}t^4$.

(b) Find $G(1)$. [1]

(c) (i) Find $G'(t)$. [2]

(ii) Hence, show that $G'(1)=\mathrm{E}(M)$. [3]

**(This question continues on the following page)**

**(Question 1 continued)**

A bag contains two red balls and three yellow balls.

Two balls are selected at random without replacement from the bag.

The random variable $X$ denotes the total number of red balls selected.

The probability distribution of $X$ can be represented by the polynomial function, $G_X$, where

$$G_X(t) = \sum_{x=0}^{2} \mathrm{P}(X = x)t^x .$$

(d) Show that $G_X(t) = \frac{3}{10} + \frac{3}{5}t + \frac{1}{10}t^2$, making it clear how the coefficients of $G_X(t)$ have been determined. [5]

An unbiased coin and a biased coin are tossed.

The probability of obtaining a tail on the biased coin is $p$.

The random variable $Y$ denotes the total number of tails obtained from tossing both coins.

The probability distribution of $Y$ can be represented by the polynomial function, $G_Y$, where

$$G_Y(t) = \sum_{y=0}^{2} \mathrm{P}(Y = y)t^y .$$

(e) Given that the coefficient of $t^2$ in $G_Y(t)$ is $\frac{1}{3}$, find

(i) the value of $p$; [2]

(ii) an expression for $G_Y(t)$. [4]

The random variable $Z$ denotes the sum of the total number of red balls selected, $X$, and the total number of tails obtained from tossing both coins, $Y$.

The probability distribution of $Z$ can be represented by the function, $G_Z$, where

$$G_Z(t) = G_X(t)\,G_Y(t) .$$

(f) For random variable $Z$, it can be shown that $G_Z'(1) = \mathrm{E}(Z)$.

Use this result to find $\mathrm{E}(Z)$. [4]

**2.** [Maximum mark: 32]

**Informally, the curvature of a curve can be thought of as the amount by which the curve deviates from being a straight line. In this question, you will investigate the curvature of a variety of functions.**

Consider any function $f$ that can be differentiated twice.

The curvature, $k$, of any function $f$ is defined by $k(x) = \dfrac{|f''(x)|}{\left(1+\left(f'(x)\right)^2\right)^{\frac{3}{2}}}$.

Consider the family of linear functions $g(x) = mx + c$, where $x \in \mathbb{R}$ and $m, c \in \mathbb{R}$.

(a) Show that $k(x) = 0$ for this family of linear functions. [2]

Consider the family of quadratic functions $h(x) = ax^2 + bx + c$ for $x \in \mathbb{R}$, where $a \in \mathbb{R}$, $a \neq 0$ and $b, c \in \mathbb{R}$.

For this family of quadratic functions, it is given that

$$k(x) = \frac{2|a|}{\left(1+(2ax+b)^2\right)^{\frac{3}{2}}} \text{ and}$$

$$k'(x) = -\frac{12a|a|(2ax+b)}{\left(1+(2ax+b)^2\right)^{\frac{5}{2}}}.$$

The maximum value of $k(x)$ is denoted by $k_{\max}$.

(b) (i) By solving $k'(x) = 0$, find the value of $x$ where $k_{\max}$ occurs.
You are **not** required to justify that this value of $x$ leads to a maximum. [1]

(ii) Determine an expression for $k_{\max}$, in terms of $a$ only. [2]

(iii) State the value of $\lim_{x \to \infty} k(x)$ and explain briefly the significance of this result. [2]

(iv) Consider the quadratic functions

$p(x) = -2x^2 + 2x - 10$ where $x \in \mathbb{R}$ and

$q(x) = 2x^2 + 5x + 25$ where $x \in \mathbb{R}$.

State which one of the following statements is true and justify your answer.

**A.** $k_{\max}$ of $p > k_{\max}$ of $q$

**B.** $k_{\max}$ of $p < k_{\max}$ of $q$

**C.** $k_{\max}$ of $p = k_{\max}$ of $q$ [2]

**(This question continues on the following page)**

**(Question 2 continued)**

Consider the function $v(x) = \ln x$, where $x \in \mathbb{R}$, $x > 0$.

For $v$, it is given that

$$k(x) = \frac{x}{\left(1 + x^2\right)^{\frac{3}{2}}} \text{ and}$$

$$k'(x) = \frac{1 - 2x^2}{\left(1 + x^2\right)^{\frac{5}{2}}}.$$

(c) (i) Determine the exact value of $x$ where $k_{\max}$ occurs.

You are **not** required to justify that this value of $x$ leads to a maximum. [2]

(ii) Show that $k_{\max} = \frac{2\sqrt{3}}{9}$. [4]

Consider the function $w(x) = \mathrm{e}^x$, where $x \in \mathbb{R}$.

For $w$, it is given that

$$k(x) = \frac{\mathrm{e}^x}{\left(1 + \mathrm{e}^{2x}\right)^{\frac{3}{2}}} \text{ and}$$

$$k'(x) = \frac{\mathrm{e}^x\left(1 - 2\mathrm{e}^{2x}\right)}{\left(1 + \mathrm{e}^{2x}\right)^{\frac{5}{2}}}.$$

(d) (i) Show that $k_{\max} = \frac{2\sqrt{3}}{9}$. [5]

(ii) Suggest a reason why $v$ and $w$ have the same $k_{\max}$. [1]

Consider a family of curves $y = \sqrt{r^2 - x^2}$, where $-r < x < r$, $y > 0$ and $r$ is a positive constant.

(e) (i) Show that $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = -\frac{r^2}{y^3}$. [6]

(ii) Hence, show that the curvature, $k$, is constant for this family of curves. [5]