2225 – 7116M

Diploma Programme
Programme du diplôme
Programa del Diploma

Markscheme

May 2025

Mathematics: analysis and approaches

Higher level

Paper 1

23 pages

International Baccalaureate
Baccalauréat International
Bachillerato Internacional

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a correct **Method**.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

R Marks awarded for clear **Reasoning**.

AG Answer given in the question and so no marks are awarded.

FT Follow through. The practice of awarding marks, despite candidate errors in previous parts, for their correct methods/answers using incorrect results.

Using the markscheme

1 General

Award marks using the annotations as noted in the markscheme *eg* ***M1, A2.***

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is generally not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *e.g.* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*e.g.* substitution into a formula) and ***A1*** for using the **correct** values.
- Where there are two or more ***A*** marks on the same line, they may be awarded independently; so if the first value is incorrect, but the next two are correct, award ***A0A1A1***.
- Where the markscheme specifies ***A3***, ***M2*** *etc*., do **not** split the marks, unless there is a note.
- The response to a "show that" question does not need to restate the ***AG*** line, unless a **Note** makes this explicit in the markscheme.
- Once a correct answer to a question or part question is seen, ignore further working even if this working is incorrect and/or suggests a misunderstanding of the question. This will encourage a uniform approach to marking, with less examiner discretion. Although some candidates may be advantaged for that specific question item, it is likely that these candidates will lose marks elsewhere too.
- An exception to the previous rule is when an incorrect answer from further working is used **in a subsequent part**. For example, when a correct exact value is followed by an incorrect decimal approximation in the first part and this approximation is then used in the second part. In this situation, award ***FT*** marks as appropriate but do not award the final ***A1*** in the first part.

Examples:

	Correct answer seen	Further working seen	Any FT issues?	Action
1.	$8\sqrt{2}$	5.65685... *(incorrect decimal value)*	No. Last part in question.	Award ***A1*** for the final mark *(condone the incorrect further working)*
2.	$\frac{35}{72}$	0.468111… (incorrect decimal value)	Yes. Value is used in subsequent parts.	Award ***A0*** for the final mark *(and full **FT** is available in subsequent parts)*

3 Implied marks

Implied marks appear in **brackets e.g. *(M1)***,and can only be awarded if **correct** work is seen or implied by subsequent working/answer.

4 Follow through marks (only applied after an error is made)

Follow through (***FT***) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s) (e.g. incorrect value from part (a) used in part (d) or incorrect value from part (c)(i) used in part (c)(ii)). Usually, to award ***FT*** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part. However, if all the marks awarded in a subsequent part are for the answer or are implied, then ***FT*** marks should be awarded for *their* correct answer, even when working is not present.

For example: following an incorrect answer to part (a) that is used in subsequent parts, where the markscheme for the subsequent part is ***(M1)A1***, it is possible to award full marks for *their* correct answer, **without working being seen.** For longer questions where all but the answer marks are implied this rule applies but may be overwritten by a **Note** in the Markscheme.

- Within a question part, once an **error** is made, no further ***A*** marks can be awarded for work which uses the error, but ***M*** marks may be awarded if appropriate.

- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks, by reflecting on what each mark is for and how that maps to the simplified version.

- If the error leads to an inappropriate value (*e.g.* probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).

- The markscheme may use the word “their” in a description, to indicate that candidates may be using an incorrect value.

- If the candidate’s answer to the initial question clearly contradicts information given in the question, it is not appropriate to award any ***FT*** marks in the subsequent parts. This includes when candidates fail to complete a “show that” question correctly, and then in subsequent parts use their incorrect answer rather than the given value.

- Exceptions to these ***FT*** rules will be explicitly noted on the markscheme.

- If a candidate makes an error in one part but gets the correct answer(s) to subsequent part(s), award marks as appropriate, unless the command term was “Hence”.

5 Mis-read

If a candidate incorrectly copies values or information from the question, this is a mis-read (***MR***). A candidate should be penalized only once for a particular misread. Use the ***MR*** stamp to indicate that this has been a misread and do not award the first mark, even if this is an ***M*** mark, but award all others as appropriate.

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*e.g.* probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- Miscopying of candidates' own work does **not** constitute a misread, it is an error.
- If a candidate uses a correct answer, to a "show that" question, to a higher degree of accuracy than given in the question, this is NOT a misread and full marks may be scored in the subsequent part.
- ***MR*** can only be applied when work is seen. For calculator questions with no working and incorrect answers, examiners should **not** infer that values were read incorrectly.

6 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If the command term is 'Hence' and not 'Hence or otherwise' then alternative methods are not permitted unless covered by a note in the mark scheme.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for parts of questions are indicated by **EITHER** . . . **OR**.

7 Alternative forms

Unless the question specifies otherwise, **accept** equivalent forms.

- As this is an international examination, accept all alternative forms of **notation** for example 1.9 and 1,9 or 1000 and 1,000 and 1.000.
- Do not accept final answers written using calculator notation. However, ***M*** marks and intermediate ***A*** marks can be scored, when presented using calculator notation, provided the evidence clearly reflects the demand of the mark.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, some **equivalent** answers will generally appear in brackets. Not all equivalent notations/answers/methods will be presented in the markscheme and examiners are asked to apply appropriate discretion to judge if the candidate work is equivalent.

8 Format and accuracy of answers

If the level of accuracy is specified in the question, a mark will be linked to giving the answer to the required accuracy. If the level of accuracy is not stated in the question, the general rule applies to final answers: *unless otherwise stated in the question all numerical answers must be given exactly or correct to three significant figures.*

Where values are used in subsequent parts, the markscheme will generally use the exact value, however candidates may also use the correct answer to 3 sf in subsequent parts. The markscheme will often explicitly include the subsequent values that come "*from the use of 3 sf values*".

Simplification of final answers: Candidates are advised to give final answers using good mathematical form. In general, for an ***A*** mark to be awarded, arithmetic should be completed, and any values that lead to integers should be simplified; for example, $\sqrt{\frac{25}{4}}$ should be written as $\frac{5}{2}$. An exception to this is simplifying fractions, where lowest form is not required (although the numerator and the denominator must be integers); for example, $\frac{10}{4}$ may be left in this form or written as $\frac{5}{2}$. However, $\frac{10}{5}$ should be written as 2, as it simplifies to an integer.

Algebraic expressions should be simplified by completing any operations such as addition and multiplication, e.g. $4e^{2x} \times e^{3x}$ should be simplified to $4e^{5x}$, and $4e^{2x} \times e^{3x} - e^{4x} \times e^{x}$ should be simplified to $3e^{5x}$. Unless specified in the question, expressions do not need to be factorized, nor do factorized expressions need to be expanded, so $x(x+1)$ and $x^2 + x$ are both acceptable.

Please note: intermediate ***A*** marks do NOT need to be simplified.

9 Calculators

No calculator is allowed. The use of any calculator on this paper is malpractice and will result in no grade awarded. If you see work that suggests a candidate has used any calculator, please follow the procedures for malpractice.

10 Presentation of candidate work

Crossed out work: If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work unless an explicit note from the candidate indicates that they would like the work to be marked.

More than one solution: Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise. If the layout of the responses makes it difficult to judge, examiners should apply appropriate discretion to judge which is "first".

Section A

1. (a) correct application of $\log_a xy = \log_a x + \log_a y$ or $\log_a x^m = m\log_a x$ ***(M1)***

correct expression in terms of $\log_{10} 2$ AND $\log_{10} 3$ (arguments must be 2 and 3) ***(A1)***

$3\log_{10} 2 + \log_{10} 3$ OR $\log_{10} 2 + \log_{10} 2 + \log_{10} 2 + \log_{10} 3$

$3p + q$ ***A1***

[3 marks]

(b) $\left(\log_3 8 =\right) \dfrac{\log_{10} 8}{\log_{10} 3} \left(= \dfrac{3\log_{10} 2}{\log_{10} 3}\right)$ ***(A1)***

$= \dfrac{3p}{q}$ ***A1***

[2 marks]

Total [5 marks]

2. (a) $f(-3) = -1$ ***A1***

[1 mark]

(b) $-3 \le x \le 5$ ***A1***

Note: Award ***A1*** for answers using interval notation $[-3,\ 5]$.

[1 mark]

(c) $\left(f^{-1}(2x-7) = -3 \Rightarrow\right) 2x - 7 = f(-3)$ **OR** $f^{-1}(-1) = -3$ ***(M1)***

$2x - 7 = -1$ ***(A1)***

$x = 3$ ***A1***

[3 marks]

Total [5 marks]

3. recognizing to use $\cos 2\theta = 2\cos^2\theta - 1$ ***(M1)***

$2\left(2\cos^2\theta - 1\right) - 5\cos\theta + 2 \;\; (=0)$ ***A1***

$4\cos^2\theta - 5\cos\theta \;\; (=0)$

choosing an appropriate method to solve their quadratic equation ***(M1)***

$\cos\theta(4\cos\theta - 5)$ **OR** $\dfrac{5 \pm \sqrt{(-5)^2 - 4\times4\times0}}{2\times4}$ ***(A1)***

$\cos\theta = 0$

$\theta = \dfrac{3\pi}{2}$ ***A1***

Note: Do not award final ***A1*** if any extra solutions given.

[5 marks]

4. (a) equating $y = mx - 3$ and $y = x^2 - x - 1$ ***M1***

$x^2 - x - 1 - (mx - 3) = 0$ **OR** $x^2 - mx - x + 2 = 0$ ***A1***

$x^2 - (m+1)x + 2 = 0$ ***AG***

[2 marks]

(b) **METHOD 1 (discriminant)**

correct substitution into discriminant (do not award if seen only in quadratic formula) ***(A1)***

$\left(-(m+1)\right)^2 - 4(1)(2)$ **OR** $(m+1)^2 - 4(1)(2)$ ***A1***

discriminant equals 0 (seen anywhere) ***A1***

$(m+1)^2 = 8$ **OR** $m^2 + 2m - 7 = 0$

$(m =)\;\; -1 - 2\sqrt{2}\,,\;\; -1 + 2\sqrt{2}$ ***A1A1***

METHOD 2 (derivative)

$\dfrac{\mathrm{d}y}{\mathrm{d}x} = 2x - 1$ ***A1***

substituting $m = 2x - 1$ or $x = \dfrac{m+1}{2}$ into ***AG*** from part (a) ***(A1)***

$x^2 - (2x - 1 + 1)x + 2 = 0$ **OR** $x^2 - 2x^2 + 2 = 0$ **OR** $\left(\dfrac{m+1}{2}\right)^2 - (m+1)\dfrac{m+1}{2} + 2 = 0$

$x = \pm\sqrt{2}$ **OR** $-m^2 - 2m + 7 = 0$ **OR** $m^2 + 2m - 7 = 0$ ***A1***

$(m =)\;\; 2\sqrt{2} - 1\,,\;\; -2\sqrt{2} - 1$ ***A1A1***

[5 marks]

Total [7 marks]

5. (a) recognizing constant change from Y to X and/or comparing the two distributions

$b-7=22-19$ **OR** $b=7+(22-19)$ **OR** $\frac{b-7}{a}=\frac{22-19}{a}$ ***(M1)***

$b=10$ ***A1***

[2 marks]

(b) $\left(\mathrm{P}(7-a<X<7+a)=\right)\ 0.68$ ***A1***

[1 mark]

(c) **EITHER**

recognizing that 22 is one standard deviation above the mean ***(M1)***

OR

recognizing symmetry of the normal curve and total area $=1$ ***(M1)***

THEN

0.16 or 0.34 (or equivalent) seen in correct sketch or probability statement ***(A1)***

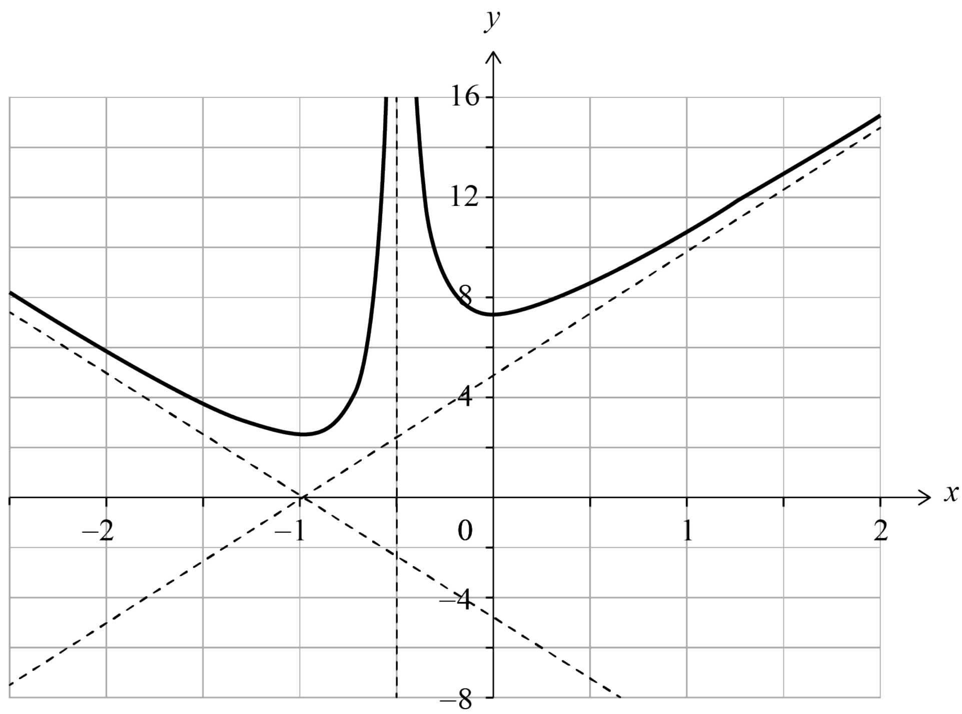
$\left(\mathrm{P}(Y<22)=\right)1-\frac{0.32}{2}$ **OR** $0.5+0.34$ **OR** $0.68+0.16$

0.84 ***A1***

[3 marks]

Total [6 marks]

6. (a)



right branch with minimum in approximately correct position **AND** showing asymptotic behaviour to $y = 5x + 5$ ***A1***

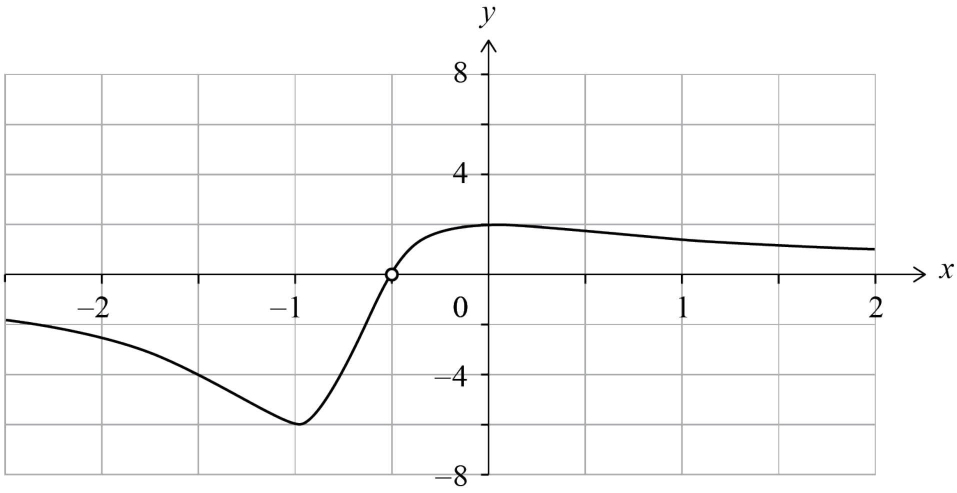
asymptotic behaviour on both branches at $x = -\frac{1}{2}$ ***A1***

left branch reflected in x-axis with minimum in approximately correct position ***M1***

oblique asymptote ($y = -5x - 5$) drawn in approximately correct position (equation is not required) **AND** left branch with correct asymptotic behaviour ***A1***

[4 marks]

(b)



axes intercepts in approximately correct positions ***A1***

local extrema in approximately correct positions ***A1***

asymptotic behavior to $y = 0$ on both sides **OR** asymptote at $y = 0$ indicated on sketch (by line or label etc. equation not required) ***A1***

[3 marks]

Total [7 marks]

7. $x^2 = r^4 - y^4$ (seen anywhere) (*A1*)

substituting their expression for x^2 into $\int_a^b \pi x^2 \mathrm{d}y$ ***(M1)***

$\pi\int_{-r}^{r}\left(\mathrm{r}^4 - y^4\right)\mathrm{d}y$ **OR** $2\pi\int_{0}^{r}\left(\mathrm{r}^4 - y^4\right)\mathrm{d}y$ (*A1*)

$= \pi\left[r^4 y - \frac{y^5}{5}\right]_{-r}^{r}$ **OR** $2\pi\left[r^4 y - \frac{y^5}{5}\right]_{0}^{r}$ ***A1***

> **Note:** Award ***A1*** for $r^4 y - \frac{y^5}{5}$ seen.

$= \pi\left(\left(r^5 - \frac{r^5}{5}\right) - \left(-r^5 + \frac{r^5}{5}\right)\right)$ **OR** $2\pi\left(r^5 - \frac{r^5}{5}\right)$ ***(A1)***

$= \frac{8}{5}\pi\mathrm{r}^5$ ***A1***

Total [6 marks]

8. (a) attempt to find modulus **OR** argument of z_1 ***(M1)***

$r = \sqrt{12}\ \left(= 2\sqrt{3}\right)$ or $\theta = -\frac{\pi}{3}$ ***A1***

$z_1 = \sqrt{12}\mathrm{e}^{-\mathrm{i}\frac{\pi}{3}}\left(= 2\sqrt{3}\mathrm{e}^{-\mathrm{i}\frac{\pi}{3}}\right)$ ***A1***

[3 marks]

(b) $\left(\frac{z_2}{z_1} = \frac{2\sqrt{3}\mathrm{e}^{\mathrm{i}\frac{5\pi}{6}}}{2\sqrt{3}\mathrm{e}^{-\mathrm{i}\frac{\pi}{3}}} = \right)\mathrm{e}^{\mathrm{i}\frac{7\pi}{6}}\left(= \mathrm{e}^{-\mathrm{i}\frac{5\pi}{6}}\right)$ ***(A1)***

recognizing three equivalent arguments ***(M1)***

eg $\frac{z_2}{z_1} = \mathrm{e}^{\mathrm{i}\left(-\frac{5\pi}{6}+2\pi k\right)}$ **OR** $\frac{z_2}{z_1} = \mathrm{e}^{-\frac{5\pi}{6}\mathrm{i}}, \mathrm{e}^{-\frac{17\pi}{6}\mathrm{i}}, \mathrm{e}^{\frac{7\pi}{6}\mathrm{i}}$

attempt to find a root using de Moivre's theorem ***(M1)***

$\mathrm{e}^{\mathrm{i}\left(-\frac{5\pi}{18}+\frac{2\pi k}{3}\right)}\left(= \mathrm{e}^{\mathrm{i}\left(\frac{(12k\text{-}5)\pi}{18}\right)} = \mathrm{e}^{\mathrm{i}\left(\frac{(12k+7)\pi}{18}\right)}\right)$ ***(A1)***

$\mathrm{e}^{-\mathrm{i}\frac{17\pi}{18}}, \mathrm{e}^{-\mathrm{i}\frac{5\pi}{18}}, \mathrm{e}^{\mathrm{i}\frac{7\pi}{18}}$ ***A1***

[5 marks]

Total [8 marks]

9. METHOD 1

$\lim\limits_{x \to 0} \dfrac{x \sin x}{1 - \cos x} = \dfrac{0}{0}$ indeterminate form, attempt to apply l'Hôpital's rule ***R1***

Note: Subsequent marks are independent of this ***R*** mark.

$= \lim\limits_{x \to 0} \dfrac{\sin x + x \cos x}{\sin x} \left(= \lim\limits_{x \to 0} \left(1 + \dfrac{x \cos x}{\sin x} \right) \right)$ ***A1***

EITHER

Applying l'Hôpital's rule again (since $\lim\limits_{x \to 0} \dfrac{\sin x + x \cos x}{\sin x} = \dfrac{0}{0}$) ***(M1)***

Note: Award ***(M0)*** if their limit is not the indeterminate form $\dfrac{0}{0}$.

$\lim\limits_{x \to 0} \dfrac{\cos x + \cos x - x \sin x}{\cos x}$ ***A1***

OR

$= \lim\limits_{x \to 0} \left(1 + \dfrac{\cos x}{\left(\dfrac{\sin x}{x} \right)} \right)$ ***A1***

recognizing fundamental trig limit $\lim\limits_{x \to 0} \dfrac{\sin x}{x} = 1$ ***R1***

THEN

substituting $x = 0$ ***(M1)***

$\left(\lim\limits_{x \to 0} \dfrac{x \sin x}{1 - \cos x} \right) = 2$ ***A1***

continued…

Question 9 continued

METHOD 2

multiplying numerator and denominator by $1+\cos x$ ***M1***

$\lim\limits_{x\to 0}\left(\frac{x\sin x}{1-\cos x}\times\frac{1+\cos x}{1+\cos x}\right)$

$\lim\limits_{x\to 0}\left(\frac{x\sin x(1+\cos x)}{1-\cos^2 x}\right)$

$\lim\limits_{x\to 0}\left(\frac{x(1+\cos x)}{\sin x}\right)$ ***A1***

$=\lim\limits_{x\to 0}\left(\frac{1+\cos x}{\left(\frac{\sin x}{x}\right)}\right)$ ***(A1)***

recognizing fundamental trig limit $\lim\limits_{x\to 0}\frac{\sin x}{x}=1$ OR $\mathbf{\lim\limits_{x\to 0}\frac{x}{\sin x}=1}$ ***(R1)***

substituting $x=0$ ***(M1)***

$\left(\lim\limits_{x\to 0}\frac{x\sin x}{1-\cos x}\right)=2$ ***A1***

continued…

Question 9 continued

METHOD 3

expressing either $x\sin x$ or $1-\cos x$ as a Maclaurin series with at least two terms ***(M1)***

$$\lim_{x\to 0}\left(\frac{x\left(x-\frac{x^3}{3!}\left(+\frac{x^5}{5!}...\right)\right)}{\frac{x^2}{2!}-\frac{x^4}{4!}\left(+\frac{x^6}{6!}+...\right)}\right)$$ ***A1***

recognizing numerator and denominator are multiples of x^2 ***(M1)***

$$\lim_{x\to 0}\left(\frac{1-\frac{x^2}{3!}\left(+\frac{x^4}{5!}-...\right)}{\frac{1}{2!}-\frac{x^2}{4!}\left(+\frac{x^4}{6!}+...\right)}\right)$$ ***A1***

substituting $x=0$ ***(M1)***

$$\left(\lim_{x\to 0}\frac{x\sin x}{1-\cos x}\right)=2$$ ***A1***

METHOD 4
Expressing in terms of half angles ***M1***

$$\lim_{x\to 0}\frac{2x\sin\frac{x}{2}\cos\frac{x}{2}}{2\sin^2\frac{x}{2}}$$ ***A1A1***

$$=\lim_{x\to 0}\frac{x\cos\frac{x}{2}}{\sin\frac{x}{2}}$$

$$=2\lim_{x\to 0}\frac{\frac{x}{2}\cos\frac{x}{2}}{\sin\frac{x}{2}}$$ ***(M1)***

Using $\lim_{x\to 0}\frac{x}{\tan x}=1$ ***M1***

=2 ***A1***

[6 marks]

Section B

10. (a) (i) **METHOD 1**

attempt to equate differences of consecutive terms ***(M1)***

$(3-2k)-(k-5)=(5k+3)-(3-2k)$ **OR** $(k-5)-(3-2k)=(3-2k)-(5k+3)$ ***A1***

$8-3k=7k$

$(k=)\ \frac{4}{5}$ ***A1***

METHOD 2 (system of equations)

<u>TWO</u> correct equations involving k and d ***A1***

$k-5+d=3-2k$ OR $3-2k+d=5k+3$ OR $k-5+2d=5k+3$

OR $\frac{3}{2}\left(2\left(k-5\right)+2d\right)=k-5+3-2k+5k+3$ (or equivalent)

valid attempt to solve their system of equations using substitution or elimination ***(M1)***

$\left(d=5.6\right)$

$(k=)\ \frac{4}{5}$ ***A1***

METHOD 3 (in terms of k)

$\frac{3}{2}\left(k-5+5k+3\right)=k-5+3-2k+5k+3$ (or equivalent) ***A1***

combining like terms ***(M1)***

$9k-3=4k+1$ **OR** $5k=4$ (or equivalent)

$(k=)\ \frac{4}{5}$ ***A1***

METHOD 4 (arithmetic mean)

attempt to find mean of u_1 and u_3 ***(M1)***

$\frac{(k-5)+(5k+3)}{2}=3-2k$ ***A1***

$3k-1=3-2k$

$(k=)\ \frac{4}{5}$ ***A1***

(ii) substituting their value of k into expression for u_3 ***(A1)***

$\left(u_3=\right)\ 5\times\frac{4}{5}+3$

$=7$ ***A1***

[5 marks]

continued…

Question 10 continued

(b) (i) substituting $k = 12$ into u_1, u_2 or u_3 ***(M1)***

$(u_1 =)\ 7$ **AND** $(u_2 =)\ -21$ **AND** $(u_3 =)\ 63$ ***(A1)***

$\left(r =\right) \frac{-21}{7} = \frac{63}{-21}\ (= -3)$ **OR** $(-21)^2 = 7 \times 63$ **OR** $r = -3$ ***R1***

u_1, u_2 and u_3 are in geometric sequence ***AG***

(ii) since $|r| \geq 1$ (accept $|r| > 1$ or $r < -1$) ***R1***

hence not convergent ***AG***

[4 marks]

(c) (i) attempts to find ratios, in terms of k, of consecutive terms and equating ***(M1)***

$\frac{(3-2k)}{(k-5)} = \frac{(5k+3)}{(3-2k)}$ **OR** $(3-2k)^2 = (k-5)(5k+3)$ (or equivalent)

$9 - 12k + 4k^2 = 5k^2 - 22k - 15$ ***A1***

Note: Award ***A1*** for correct expansion of all brackets leading to given result.

$k^2 - 10k - 24 = 0$ ***AG***

(ii) recognizing need to factorize, complete the square or substitute into quadratic formula ***(M1)***

$(k+2)(k-12)\left(=0\right)$ **OR** $(k-5)^2 - 49\ (= 0)$ **OR** $k = \frac{10 \pm \sqrt{196}}{2}$

$k = -2$ (accept $k = -2$ **and** $k = 12$) ***A1***

substituting their value of k (other than $k = 12$) to find u_1, u_2 or u_3 ***(M1)***

$(u_1 =)\ -7$ **AND** $(u_2 =)\ 7$ **AND** $(u_3 =)\ -7$ ***A1***

(iii) $(S_{2m} =)\ 0$ ***A1***

[7 marks]

Total [16 marks]

11. (a) $\overrightarrow{\mathrm{BC}} = \begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix}$ **OR** $\overrightarrow{\mathrm{BA}} = \begin{pmatrix} 4 \\ 2 \\ -2 \end{pmatrix}$ ***(A1)***

$(3,\ 0,\ 2)$ ***A1***

[2 marks]

(b) **EITHER**

recognizing to find midpoint of $[\mathrm{AC}]$ or $[\mathrm{BD}]$ ***(M1)***

$\left(\frac{-1+1}{2}, \frac{-2-4}{2}, \frac{4-0}{2}\right)$ **OR** $\left(\frac{-3+3}{2}, \frac{-6+0}{2}, \frac{2+2}{2}\right)$

OR

Let $\mathrm{E}\,(a,\ b,\ c)$

$\overrightarrow{\mathrm{AE}} = \overrightarrow{\mathrm{EC}} \Rightarrow \begin{pmatrix} a-1 \\ b+4 \\ c \end{pmatrix} = \begin{pmatrix} -1-a \\ -2-b \\ 4-c \end{pmatrix}$ **OR** $\overrightarrow{\mathrm{AE}} = \frac{1}{2}\overrightarrow{\mathrm{AC}} \Rightarrow \begin{pmatrix} a-1 \\ b+4 \\ c \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -2 \\ 2 \\ 4 \end{pmatrix}$ ***(M1)***

THEN

$(0,\ -3,\ 2)$ ***A1***

[2 marks]

continued…

Question 11 continued

(c) (i) $\vec{AB} = \begin{pmatrix} -4 \\ -2 \\ 2 \end{pmatrix}$ **AND** $\vec{AD} = \begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix}$ ***A1***

Note: Award ***A1*** for finding both vectors, one of which may have been seen in (a).

$$\begin{pmatrix} -4 \\ -2 \\ 2 \end{pmatrix} \times \begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix} = \begin{pmatrix} -12 \\ 12 \\ -12 \end{pmatrix}$$

$m = 12$ ***A1***

(ii) $\sqrt{(-12)^2 + 12^2 + (-12)^2}$ **OR** $12\sqrt{(-1)^2 + 1^2 + (-1)^2}$ ***(A1)***

$= 12\sqrt{3}$ ***A1***

[4 marks]

(d) substituting cross product and a point into the equation of a plane ***(M1)***

$-x + y - z = -5$ ***A1***

[2 marks]

continued…

Question 11 continued

(e) let a normal to Π_1 be $\boldsymbol{n}_1$ and a normal to Π_2 be $\boldsymbol{n}_2$

Eg. $\boldsymbol{n}_1 = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$ and $\boldsymbol{n}_2 = \begin{pmatrix} 5 \\ 1 \\ -7 \end{pmatrix}$

recognizing angle between $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ is required ***(M1)***

$|\boldsymbol{n}_2| = \sqrt{75}$ ***A1***

$\cos\theta = \dfrac{-1\times5+1\times1+(-1)\times(-7)}{\sqrt{3}\times\sqrt{75}} \quad \left(= \dfrac{3}{\sqrt{3}\times\sqrt{75}}\right)$ ***A1***

> **Note:** The negative of this may be found but to obtain the final ***A1*** there needs to be a recognition that this obtains the obtuse angle for the minus to be dropped.

$\cos\theta = \dfrac{1}{5}$ ***AG***

[3 marks]

(f) attempt to substitute normal to Π_1 and their E into vector equation of line ***(M1)***

$(\boldsymbol{r} =) \begin{pmatrix} 0 \\ -3 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$ ***A1***

substitutes for λ into Π_2 ***(M1)***

$5(-\lambda)+(-3+\lambda)-7(2-\lambda)=1$

$\lambda = 6$ ***A1***

$(-6, 3, -4)$ ***A1***

[5 marks]

Total [18 marks]

12. (a) substituting into modulus formula with either correct real part or correct imaginary part ***(M1)***

$\sqrt{(x-2)^2+(y-1)^2}\ \ (=3)$ ***(A1)***

$x^2-4x+4+y^2-2y+1=9$ ***A1***

$x^2+y^2-4x-2y-4=0$ ***AG***

[3 marks]

(b) **METHOD 1**

recognizing the need to express $\frac{z+p}{z-1}$ in the form $a+\mathrm{i}b$ ***(M1)***

$\frac{z+p}{z-1}\times\frac{z^*-1}{z^*-1}$ **OR** $\frac{z+p}{z-1}\times\frac{(x-y\mathrm{i})-1}{(x-y\mathrm{i})-1}$ **OR** $\frac{(x+p)+\mathrm{i}y}{(x-1)+\mathrm{i}y}\times\frac{(x-1)-\mathrm{i}y}{(x-1)-\mathrm{i}y}$ ***A1***

expanding their $\frac{z+p}{z-1}\times\frac{z^*-1}{z^*-1}$ ***(M1)***

$\frac{zz^*-z+pz^*-p}{zz^*-z-z^*+1}$ **OR** $\frac{x^2+y^2-x-y\mathrm{i}+px-py\mathrm{i}-p}{x^2+y^2-x-y\mathrm{i}-x+y\mathrm{i}+1}$

OR $\frac{(x+p)(x-1)+y^2+((x-1)y-(x+p)y)\mathrm{i}}{(x-1)^2+y^2}$

$\frac{x^2+y^2+(p-1)x-p-(p+1)y\mathrm{i}}{x^2+y^2-2x+1}$ (or equivalent) (seen anywhere) ***A1***

$\left(\Rightarrow \arg\left(\frac{z+p}{z-1}\right)=\right)\ \arctan\left(\frac{-(p+1)y}{x^2+y^2+(p-1)x-p}\right)$ ***(A1)***

equating their $\arg\left(\frac{z+p}{z-1}\right)$ to $\frac{\pi}{4}$ **OR** their expression for $\frac{z+p}{z-1}$ in x and y to 1
OR equating real and imaginary parts ***(M1)***

$\boldsymbol{\arctan\left(\frac{-(p+1)y}{x^2+y^2+(p-1)x-p}\right)=\frac{\pi}{4}}$ **OR** $\frac{-(p+1)y}{x^2+y^2+(p-1)x-p}=1$

$-(p+1)y=x^2+y^2+(p-1)x-p$ ***A1***

$x^2+y^2+(p-1)x+(p+1)y-p=0$ ***AG***

continued…

Question 12 continued

METHOD 2

$\arg\left(\frac{z+p}{z-1}\right) = \arg(z+p) - \arg(z-1)$ ***(M1)***

$= \arg(x+p+y\mathrm{i}) - \arg(x-1+y\mathrm{i})$ ***A1***

$= \arctan\left(\frac{y}{x+p}\right) - \arctan\left(\frac{y}{x-1}\right)$ ***A1***

attempt to use compound angle formula for tan ***(M1)***

$$\tan\left(\arctan\left(\frac{y}{x+p}\right) - \arctan\left(\frac{y}{x-1}\right)\right) = \frac{\tan\left(\arctan\left(\frac{y}{x+p}\right)\right) - \tan\left(\arctan\left(\frac{y}{x-1}\right)\right)}{1+\tan\left(\arctan\left(\frac{y}{x+p}\right)\right)\times\tan\left(\arctan\left(\frac{y}{x-1}\right)\right)}$$

$$= \frac{\frac{y}{x+p} - \frac{y}{x-1}}{1+\frac{y}{x+p}\times\frac{y}{x-1}}$$ ***A1***

equating their expression for $\tan\left(\arctan\left(\frac{y}{x+p}\right) - \arctan\left(\frac{y}{x-1}\right)\right)$ to $\tan\frac{\pi}{4}$ ***(M1)***

$$\frac{\frac{y}{x+p} - \frac{y}{x-1}}{1+\frac{y}{x+p}\times\frac{y}{x-1}} = \tan\frac{\pi}{4} \quad \left(\Rightarrow \frac{(x-1)y-(x+p)y}{(x+p)(x-1)+y^2} = 1\right)$$

$(x+p)(x-1)+y^2 = -y-py$ ***A1***

$x^2 - x + px - p + y^2 = -y - py$

$x^2 + y^2 + (p-1)x + (p+1)y - p = 0$ ***AG***

[7 marks]

continued…

Question 12 continued

(c) $\left(\arg\left(\frac{z+4}{z-1}\right)=\frac{\pi}{4}\Rightarrow\right)$ $x^2+y^2+3x+5y-4=0$ (seen anywhere) ***A1***

solving simultaneously their equations in x and y ***(M1)***

$7x+7y=0$ $(\Rightarrow x=-y)$ (or equivalent) ***A1***

substituting to obtain an equation in either x or y ***(M1)***

$x^2-x-2=0$ **OR** $y^2+y-2=0$

solving for either x or y ***(M1)***

$x=-1$ and $x=2$ **OR** $y=-2$ and $y=1$ ***A1***

$(z_1=)$ $-1+\mathrm{i}$ **AND** $(z_2=)$ $2-2\mathrm{i}$ (or equivalent) ***A1***

Note: Accept $\boldsymbol{x=-1,\ y=1}$ **AND** $\boldsymbol{x=2,\ y=-2}$.

[7 marks]

(d) let z_1, z_2, z_3 and z_4 be roots of $z^4+ax^3+bz^2+cz+d=0$

EITHER

$a=-(z_1+z_2+z_3+z_4)$ ***(A1)***

OR

expanding $(z-z_1)(z-z_2)(z-z_3)(z-z_4)$ to obtain term in z^3

$-(z_1+z_2)z^3-(z_3+z_4)z^3$ ***(A1)***

THEN

recognizing that z_3 and z_4 are conjugates of z_1 and z_2 (seen anywhere) ***(M1)***

roots are $-1+\mathrm{i}$, $-1-\mathrm{i}$, $2-2\mathrm{i}$, $2+2\mathrm{i}$ ***A1***

$a=-((-1+\mathrm{i})+(-1-\mathrm{i})+(2-2\mathrm{i})+(2+2\mathrm{i}))$

$=-2$ ***A1***

[4 marks]

Total [21 marks]